2025

Four Year Under Graduate Programme (FYUGP)

5th Semester Examination (Under NEP)
(Session 2023-24)

MATHEMATICS (Major)

Paper Code : MTMMJ MC-08

[Linear Algebra]

Full Marks : 50

Time : Two Hours

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer any *five* questions : 2×5=10

(a) Determine whether the set $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ is linearly dependent or independent in $M_{2\times 2}(\mathbb{R})$.

(b) Let V be a vector space over a field $\mathbb{F}$ having a finite basis. Prove that every basis for V contains the same number of vectors.

(c) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be linear such that $T(1,0)=(1,4)$ and $T(1,1)=(2,5)$. What is $T(2,3)$?

(d) Give an example, with explanation, of a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker(T)=\operatorname{Im}(T)$.

(e) Prove or disprove : The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

(f) Let T be an invertible linear operator on a finite dimensional vector space V. If λ is an eigenvalue of T, prove that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(g) Prove or disprove : If x, y and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$, then $y=z$.

(h) Let V be a non-zero finite dimensional inner product space. Does V have an orthonormal basis? Justify your answer.

Group - B

Answer any *four* questions : 10×4=40

2. (a) Let V be a finite-dimensional vector space, and let $B=\{v_1, v_2, \dots, v_n\}$ be a subset of V. Prove that B is a basis for V if and only if every $v \in V$ can be uniquely expressed as a linear combination of vectors of B.

(b) Let W be a subspace of a finite-dimensional vector space V. Prove that W is finite-dimensional and $\dim W \le \dim V$. Moreover, if $\dim W = \dim V$, prove that $W = V$. 5+(3+2)

3. (a) Let V and W be two vector spaces over the same field F and $T:V \to W$ be a linear transformation. Prove that $\ker T$ and $\operatorname{Im} T$ are subspaces of V and W, respectively.

(b) Let V and W be two finite-dimensional vector spaces over the same field $\mathbb{F}$ with $\dim V = \dim W$. Prove that a linear transformation $T:V \to W$ is one-to-one if and only if it is onto. 5+5

4. (a) Find the matrix representation of the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^3$, defined by

$$T(x, y) = (x-y, x, 2x+y)$$

for all $(x, y) \in \mathbb{R}^2$, with respect to the standard ordered basis for $\mathbb{R}^2$ and the basis $\{(1,1,0),(0,1,1),(2,2,3)\}$ for $\mathbb{R}^3$.

(b) Let V and W be two finite dimensional vector spaces over the same field. Prove that V is isomorphic to W if and only if $\dim V = \dim W$. 5+5

5. (a) Let T be a linear operator on a vector space V, and let λ be an eignevalue of T. Prove that a vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \in \operatorname{Im}(T - \lambda I) - \{\theta\}$.

(b) Let $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ be a linear operator defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

for all $a, b, c, d \in \mathbb{R}$. Find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

5+5

6. (a) Let T be a linear operator on a finite-dimensional vector space V such that its characteristic polynomial splits and $\lambda_1, \lambda_2, \cdots, \lambda_k$ be its distinct eigenvalues. Prove that T is diagonalizable if and only if the algebraic multiplicity of λ_i is equal to $\dim\left(E_{\lambda_i}\right)$ for each $i = 1, 2, \cdots, k$.

(b) Check whether the linear operator $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(f(x)) = f'(x)$ for all $f(x) \in P_2(\mathbb{R})$, is diagonalizable or not.

5+5

7. (a) Let V be an inner product space over $\mathbb{F}$. For all $x, y \in V$ and $c \in \mathbb{F}$, prove that

(i) $|\langle x, y\rangle| \le \|x\|\|y\|$,

(ii) $\|x + y\| \le \|x\| + \|y\|$,

(iii) if $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

(b) Let $P_2(\mathbb{R})$ be endowed with the inner product $\langle \cdot, \cdot \rangle$ defined by $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$ for all $f(x), g(x) \in P_2(\mathbb{R})$. Compute an orthonormal basis for $P_2(\mathbb{R})$.

(2+2+2)+4

FYUGP/5th Sem/25(NEP)

2025

Four Year Under Graduate Programme (FYUGP)

5th Semester Examination (Under NEP)
(Session 2023-24)

MATHEMATICS (Major)

Paper Code : MTMMJ MC-09

[Real Analysis-II]

Full Marks : 50

Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meanings.

Group - A

1. Answer any *five* questions : 2×5=10

(a) A function $f:[0,1]\to\mathbb{R}$ is defined by

$$f(x)=x^2\cos\frac{1}{x},\quad \text{if } x\neq 0$$
$$=0,\quad \text{if } x=0.$$

Show that f is a function of bounded variation on [0, 1].

(b) Evaluate $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} e^{\sqrt{1+t}}\, dt}{x^2}$.

(c) Examine the convergence of the improper integral $\int_1^{\infty} \dfrac{1}{x\left(1+x^2\right)} dx$.

(d) Find the value of $\int_0^{\infty} e^{-x^2}\, dx$.

(e) Evaluate the limit

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}.$$

(f) Show that $\int_0^{\pi/2} \dfrac{x^m}{\sin^n x} dx$ is convergent if and only if $n < 1+m$.

(g) Show that $f:[0,1]\to\mathbb{R}$ be defined by

$$f(x) = 1, \text{ if } x \text{ be rational}$$
$$= 0, \text{ if } x \text{ be irrational}$$

is not of bounded variation.

(h) If $f_n(x) = \tan^{-1} nx, x\in[0,1]$. Prove that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

Group - B

Answer any *four* questions : 10×4=40

2. (a) Let a function $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$. Prove that f is integrable on $[a,b]$ if and only if for each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P,f) - L(P,f) < \epsilon$.

(b) A function f is defined on [0, 1] by

$$f(x) = x, x\in[0,1]\cap\mathbb{Q}$$
$$= 0, x\in[0,1]-\mathbb{Q}$$

find $\int_{-0}^{1} f$ and $\int_{0}^{-1} f$. Deduce that f is not integrable on [0, 1]. 6+4

3. (a) Show that the improper integral $\int_0^{\infty} \dfrac{\sin x}{x} dx$ is convergent but $\int_0^{\infty} \left|\dfrac{\sin x}{x}\right| dx$ is not convergent.

(b) Show that the improper integral $\int_0^{\infty} \dfrac{\cos x}{1+x^2} dx$ is absolutely convergent. 6+4

P.T.O.

4. (a) For each $n \in \mathbb{N}$, $f_n(x) = nxe^{-nx^2}$, $x \in [0, 1]$. Show that the sequence $\{f_n\}$ is not uniformly convergent on [0, 1].

(b) Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ is continuous on D. If the sequence $\{f_n\}$ is uniformly convergent on D to a function f, then f is continuous on D. 5+5

5. (a) Find the Fourier series of the following function defined by

$$f(x) = \begin{cases} -x, & \text{for } -\pi < x < 0 \\ x, & \text{for } 0 < x < \pi \end{cases}$$

Hence deduce that, $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots\cdots$

(b) If f be bounded and integrable on $[-\pi, \pi]$ and if a_n, b_n are Fourier co-efficients, then prove that $\{a_n\}$ and $\{b_n\}$ form null sequences. (4+2)+4

6. (a) Let $f : [a, b] \in \mathbb{R}$. Prove that f is a function of bounded variation on [a, b] if and only if f can be expressed as the difference of two monotonically increasing function on [a, b].

(b) If $\displaystyle\int_0^\infty \frac{x^{m-1}}{1+x}dx = \frac{\pi}{\sin mx}$, show that

$$\Gamma m \cdot \Gamma(1-m) = \frac{\pi}{\sin m\pi},\ 0 < m < 1.$$ 6+4

7. (a) Prove that if a function $f : [a, b] \to \mathbb{R}$ is monotonic, then f is integrable on $[a, b]$. Also prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable on [a, b].

(b) A function f is defined on [0, 1] by $f(0) = 0$

$$f(x) = \frac{1}{2^n},\ \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}\ (n = 0, 1, 2, \ldots)$$

Show that f is integrable on [0, 1] and $\int_0^1 f = \frac{2}{3}$.

(3+3)+4

2025

Four Year Under Graduate Programme (FYUGP)

5th Semester Examination (Under NEP)
(Session 2023-24)

MATHEMATICS (Major)

Paper Code : MTMMJ MC-10

[Multivariate Calculus and Metric Spaces]

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer any *five* questions : 2×5=10

(a) Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y)=\begin{cases} y\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0), \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$$

Compute the first-order partial derivative of f at the origin.

P.T.O.

(b) Find the critical points of $f(x, y) = x^3 + y^3 - 3xy$

(c) Reverse the order of integration in $\int_0^1 dx \int_x^{\sqrt{x}} f(x, y)\,dy$

(d) Compute the line integral of $f(x, y) = \sqrt{\left(\frac{bx}{a}\right)^2 + \left(\frac{ay}{b}\right)^2}$ over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(e) Let d be a metric on a non-empty set X. Prove that the function $\bar{d} : X \times X \to \mathbb{R}$ defined by

$\bar{d}(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$ is also a metric on X.

(f) Cite an example of a metric space in which every Cauchy sequence is convergent.

(g) Consider $\mathbb{R}$ equipped with the usual/standard metric d, and $(-1, 1)$ equipped with the induced metric by d on $(-1, 1)$. Check whether the mapping $f : \mathbb{R} \to (-1, 1)$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in \mathbb{R}$, is continuous or not.

(h) Prove that every contraction mapping is uniformly continuous.

Group - B

Answer any *four* questions : 10×4=40

2. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = \begin{cases} y + x\sin\left(\frac{1}{y}\right), & y \neq 0 \\ 0 & , y = 0 \end{cases}$$

Show that $\lim_{(x, y)\to(0, 0)} f(x, y)$ and $\lim_{y\to 0} \lim_{x\to 0} f(x, y)$ exist but $\lim_{x\to 0} \lim_{y\to 0} f(x, y)$ does not exist.

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x, y, z) = \left(\frac{x^2 - y^4 - 2z}{2}, yz\right)$$

for all $(x, y, z) \in \mathbb{R}^3$. Find the derivative (Jacobian matrix) $D_f(x, y, z)$. If $f(x, y, z) = (u, v)$, find the Jacobians $\frac{\partial(u, v)}{\partial(x, y)}, \frac{\partial(u, v)}{\partial(y, z)}$ and $\frac{\partial(u, v)}{\partial(x, z)}$.

5+(2+3)

3. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Find $\frac{\partial f}{\partial x}(x, 0)$, $\frac{\partial f}{\partial y}(0, y)$, $\frac{\partial f}{\partial x}(0, y)$ and $\frac{\partial f}{\partial y}(x, 0)$. Is the scalar field f differentiable at (0, 0)? Justify your answer.

(b) Evaluate by Green's theorem

$$\int_\Gamma (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy,$$

where Γ is the circle $x^2 + y^2 = 1$ in the xy-plane described in the positive sense. (4+2)+4

4. (a) Evaluate the integral by changing the order of integration $\int_0^1 \int_x^{1/x} \frac{y^2}{(x+y)^2 \sqrt{1+y^2}}\,dy\,dx$.

(b) Compute the integral $\iiint_V \left(\frac{1}{\sqrt{x^2 + y^2}} + \frac{1}{z} \right) dx\,dy\,dz$, where V is the region in $\mathbb{R}^3$ bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 1$, and $z > 0$. 5+5

5. (a) Let $C[a,b]$ denote the set of all real-valued continuous functions defined on $[a,b]$. Define $d : C[a,b] \times C[a,b] \to \mathbb{R}$ by

$$d(f, g) = \max_{t \in [a, b]} |f(t) - g(t)| \text{ for all}$$

$f, g \in C[a,b]$. Prove that $(C[a,b], d)$ is a metric space.

(b) Let (X, d) be a metric space. Prove that an arbitrary intersection of closed sets in X is closed in X. Is it true for arbitrary union? Justify your answer. 5+(3+2)

6. (a) State Gauss Divergence theorem and apply it to evaluate the surface integral

$$\iint_S x^3 dydz + y^3 dzdx + z^3 dxdy,$$ where s is the sphere $x^2 + y^2 + z^2 = a^2$.

(b) Let $B(S)$ denote the set of all real or complex-valued functions on S, each of which is bounded. Let $B(S)$ is endowed with the metric d defined by

$$d(f, g) = \sup_{t \in S} |f(t) - g(t)|,\ f, g \in B(S).$$

Prove that $(B(S), d)$ is a complete metric space. 5+5

P.T.O.

7. (a) Let (X, d) and (Y, ρ) be metric spaces and $f, g : X \to Y$ be continuous functions. Prove that the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

(b) Let (X, d) be a metric space and let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of connected sets in X such that $\bigcap_{\alpha \in \Lambda} A_\alpha \neq \phi$. Prove that $\bigcap_{\alpha \in \Lambda} A_\alpha$ is connected.

(c) State and prove Euler's theorem for homogeneous function.

2+3+5

2025

Four Year Under Graduate Programme (FYUGP)

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MATHEMATICS (Major)

Paper Code : MTMMJ MC-11

[LPP, Game Theory and Integral Transforms-I]

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer any *five* questions : 2×5=10

(a) Find the basic solutions of the system $x_1 + 2x_3 = 1,\ x_2 + x_3 = 4$.

(b) Prove that a hyperplane is a convex set.

(c) Obtain the dual of the following L.P.P.

Minimize $Z = 3x_1 - 2x_2$

subject to $2x_1 + x_2 \leq 1$

$-x_1 + 3x_2 \geq 4$

$x_1, x_2 \geq 0$

(d) State Heaviside's expansion theorem for finding the inverse Laplace transform of a rational function.

(e) If $L\{F(t)\} = f(p)$, then prove that $L\{F'(t)\} = pf(p) - F(0)$ if $F(t)$ is continuous for $0 \leq t \leq N$ and of exponential order for $t > N$ while $F'(t)$ is sectionally continuous for $0 \leq t \leq N$.

(f) If the indicated limits exist, prove $\lim_{t\to 0} F(t) = \lim_{p\to\infty} pf(p)$ where $f(p) = L\{F(t)\}$.

(g) Find $L^{-1}\left\{\frac{p}{(p^2-1)^2}\right\}$.

(h) Evaluate $L\{t^3 \cos t\}$.

Group - B

Answer any *four* questions : 10×4=40

2. (a) Use two phase method to show that the following L.P.P. has unbounded :

Solution :

Maximize $Z = 2x_1 + 3x_2 + x_3, x_1, x_2, x_3 \geq 0$

subject to $-3x_1 + 2x_2 + 3x_3 = 8$

$-3x_1 + 4x_2 + 2x_3 = 7$ 6

(b) Evaluate $L(|\cos t|)$ 4

3. (a) Find the optimal solution of the following transportation problem :

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	3	4	6	8	8	20
O_2	2	10	0	5	8	30
O_3	7	11	20	40	3	15
O_4	1	0	9	14	16	13
b_j	40	6	8	18	6	

7

(b) Evaluate $L^{-1}\left\{\frac{2p+3}{(p+1)^2(p+2)^2}\right\}$. 3

4. (a) If x° be any feasible solution to the primal problem and v° be any feasible solution to the dual problem, then show that $cx^\circ \leq b^T v^\circ$. 5

(b) Prove that $L\{erf(t^{1/2})\} = \frac{1}{p(p+1)^{\frac{1}{2}}}$. 5

P.T.O.

5. (a) Five operators have to be assigned to five machines. The assignment costs are given below. Operator A cannot operate machine III and operator C cannot operate machine IV. Find the optimal assignment schedule.

OPERATORS \ MACHINES	I	II	III	IV	V
A	5	5	–	2	6
B	7	4	2	3	4
C	9	3	5	–	3
D	7	2	6	7	2
E	6	5	7	9	1

6

(b) Use Laplace Transform to prove that

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt = \log\left(\frac{2}{3}\right).$$

4

6. (a) Construct the dual of the following L.P.P. and solve both the primal and the dual :

Maximize $Z = 3x_1 + 4x_2$

Subject to $x_1 + x_2 \le 12$

$2x_1 + 3x_2 \le 21$

$x_1 \le 8$

$x_2 \le 6$

$x_1, x_2 \ge 0$

6

(b) Use Laplace Transform to solve the initial value problem

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = 4\sin^2 t,\ x(0) = 0,\ x'(0) = -1.$$

4

7. (a) Use dominance principle to reduce the pay off matrix and solve the game with the following pay off matrix

	B			
A	8	15	–4	–2
	19	15	17	16
	0	20	15	5

5

(b) Use Convolution theorem to show that

$$L^{-1}\left\{\frac{1}{(p+2)^2(p-2)}\right\} = \frac{1}{16}\left(e^{2t} - 4te^{-2t} - e^{-2t}\right).$$

5